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Asymmetric gamma generalized curved normal distribution

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ABSTRACT
Here we consider a new class of asymmetric mixture normal distribution and investigate some of its important properties. Location-scale extension of the proposed model is also considered and discussed the estimation of its parameters by method of maximum likelihood. Two real life data sets are considered for illustrating the usefulness of the model.

KEYWORDS
Asymmetric distributions; Characteristic function; Maximum likelihood estimation; Plurimodality; Reliability measures.

1. Introduction

The normal distribution is the basis of many statistical works and it enjoys a unique position in probability theory. It is an unavoidable tool for the analysis and interpretation of data. In many practical applications it has been observed that real life data sets are not symmetric. They exhibit some skewness, therefore do not conform to the normal distribution, which is popular and easy to be handled. Azzalini (1985) introduced a new class of distributions namely "the skew normal distribution", which is mathematically tractable and includes the normal distribution as a special case. This family of distributions is well known for modeling and analyzing skewed data. This distribution has been developed via standard normal probability density function (p.d.f) and cumulative distribution function (c.d.f) through adding a shape parameter to regulate skewness, so as to have more flexibility in fitting real life data sets.

Let $\phi(.)$ and $\Phi(.)$ be the p.d.f and c.d.f of a standard normal variate. Then a random variable X_1 is said to follow the skew normal distribution with parameter $\lambda \in R = (-\infty, \infty)$ if its probability density function (p.d.f.) $h(x;\lambda)$ is of the following form. For $x \in R$,

$$h(x;\lambda) = 2\phi(x)\,\Phi(\theta(x)), \tag{1}$$

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hereafter, we denoted a distribution with p.d.f. (1) as $SND(\lambda)$. This distribution has been studied by several authors such as Azzalini (1986), Henze (1986), Azzalini and Dalla Valle (1996), Branco and Dey (2001), Kumar and Anusree (2011, 2013, 2014a,b).

The normal and skew normal models are not adequate to describe the situations of plurimodality. To overcome this drawback Kumar and Anusree (2011) considered a new class of generalized skew normal distribution as a generalized mixture of standard normal and skew normal distributions through the following p.d.f in which $x \in R$, $\lambda \in R$ and $\beta > -1$.

$$h_1(x;\lambda,\beta) = \frac{2}{\beta+2}\phi\left(x\right)\left[1+\beta\Phi\left(\theta(x)\right)\right]. \tag{2}$$

The distribution given in (2) they termed as generalized mixture of standard normal and skew normal distributions ($GMNSND(\lambda,\alpha)$). Clearly $GMNSND(\lambda,-1)$ is $SN(-\lambda)$. In order to develop a more flexible plurimodal asymmetric normal distribution, Kumar and Anila (2023) developed a generalized form of GMNSND through the following p. d. f

$$h_2(x;\lambda,\beta,\alpha) = \frac{\phi(x)}{\beta+2}\left[2+\beta[\Phi(\alpha)]^{-1}\Phi\left(\theta(x)\right)\right]. \tag{3}$$

in which $x \in R$, $\lambda,\alpha \in R$ and $\beta > -1$. The distribution given in (3) they termed as Modified skew generalized normal distributions ($MSGND(\lambda,\alpha,\beta)$). Through the present paper we consider a generalized version of the skew normal distribution of Kumar and Anila (2017) which we call "the asymmetric gamma generalized curved normal distribution (AGGCND)".

The organization of the paper is as follows. In section 2 we present the definition and some properties of the AGGCND. In section 3 certain reliability measures such as reliability function, failure rate, and mean residual life function are derived and condition for unimodal and plurimodal situations are obtained. In section 4 a location scale extension of the AGGCND is proposed and derive its important properties such as characteristic function, mean, variance, measure of skewness and kurtosis, reliability measures etc. Further in section 5 we discuss the maximum likelihood estimation of the parameters of extended AGGCND and a real life application of the distribution is considered in section 6.

2. The asymmetric gamma generalized curved normal distribution

Here we define a new class of generalized skew normal distribution and derive some of its important properties.

Definition 2.1. A random variable X_1 is said to have a asymmetric gamma generalized curved normal distribution if its p.d.f is of the following form, in which $x \in R$, $\lambda,\alpha,\gamma \in R$, and $\beta > -1$.

$$f_1(x;\lambda,\beta,\alpha,\gamma) = \frac{\phi(x)}{\gamma+\beta}\left[\gamma+\beta[\Phi(\alpha)]^{-1}\Phi(\theta(x)),\right] \tag{4}$$

where $\theta(x) = \alpha\sqrt{1+\lambda^2} + \frac{\lambda x}{\sqrt{1+\lambda^2 x^2}}$, for convenience of notation. A distribution with

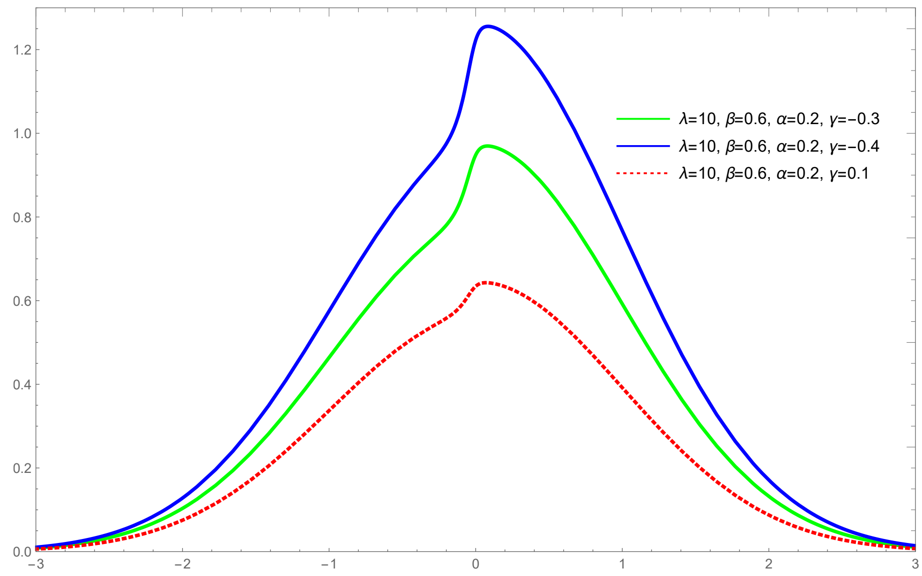

Figure 1. Probability plots of AGGCND$(\lambda, \beta, \alpha, \gamma)$ for fixed values of λ, β, α and various values of γ.

p.d.f (4) we denoted as AGGCND$(\lambda, \beta, \alpha, \gamma)$.

3. Special Cases

when

(1) $\gamma = 2$, AGGCND$(\lambda, \beta, \alpha, \gamma)$ reduces to the asymmetric curved normal distribution (ACND) of Kumar and Anila (2018).
(2) $\gamma = 0$ and $\alpha = 0$ AGGCND$(\lambda, \beta, \alpha, \gamma)$ reduces to the skew curved normal distribution (SCND) of Arellano-Valle et al. (2004).
(3) $\beta = 0$, AGGCND$(\lambda, \beta, \alpha, \gamma)$ reduces to the standard normal distribution.
(4) $\gamma = 2$ and $\alpha = 0$, AGGCND$(\lambda, \beta, \alpha, \gamma)$ reduces to the extended skew curved normal distribution (ESCND) of Kumar and Anusree (2017).

For some particular choices of α, λ and β the p.d.f. given in (4) of AGGCND(α, λ, β) is plotted in Figure 1. Now we obtain the following results which gives some structural properties of the distribution AGGCND.

Proposition 3.1. *If X_1 has AGGCND$(\lambda, \beta, \alpha, \gamma)$ then $Y_1 = -X$ has AGGCND$(-\lambda, \beta, \alpha, \gamma)$*

Proof. The p.d.f $f_1^{(1)}(z)$ of Z_1 is

$$
\begin{aligned}
f_1(z) &= f_1(-z; \lambda, \beta, \alpha, \gamma)|\frac{dx}{dz}| \\
&= \frac{\phi(-z)}{\gamma + \beta}\left[\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(-z))\right] \\
&= f_1(z; -\lambda, \beta, \alpha, \gamma)
\end{aligned}
$$

Since $\phi(.)$ is the p. d. f of standard normal variate. Hence Z_1 follows

AGGCND$(-\lambda, \beta, \alpha, \gamma)$. □

Proposition 3.2. *If X_1 has AGGCND$(\lambda, \beta, \alpha, \gamma)$ then $Z_2 = X_1^2$ has a p.d.f (5) in which $\Delta_1(z) = \Phi(\theta(z)) + \Phi(\theta(-z))$.*

Proof. The p.d.f. $f_1^{(2)}(z)$ of $Z_2 = X_1^2$ is the following, for $z > 0$.

$$\begin{aligned} f_1(z) &= f_1(\sqrt{z}, \lambda, \beta, \alpha, \gamma)|\frac{dx}{dz}| + f_1(-\sqrt{z}, \lambda, \beta, \alpha, \gamma)|\frac{dx}{dz}| \\ &= \frac{\phi(\sqrt{z})}{\gamma+\beta}\left[\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(\sqrt{z}))\right]\frac{1}{2\sqrt{y_2}} + \\ &\quad \frac{\phi(-\sqrt{z})}{\gamma+\beta}\left[\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(-\sqrt{z}))\right]\frac{1}{2\sqrt{z}} \\ &= \frac{\phi(\sqrt{z})}{2(\gamma+\beta)\sqrt{z}}\left[2\gamma + \beta[\Phi(\alpha)]^{-1}\left\{\Phi(\theta(\sqrt{z})) + \Phi(\theta(-\sqrt{z}))\right\}\right] \\ &= \left(\frac{\phi(\sqrt{y_2})}{2\sqrt{z}}\right)\frac{1}{(\gamma+\beta)}\left[2\gamma + \beta[\Phi(\alpha)]^{-1}\Delta(\sqrt{z})\right] \qquad (5) \end{aligned}$$

□

Proposition 3.3. *If X_1 has AGGCND$(\lambda, \beta, \alpha, \gamma)$ then $Z_3 = |X_1|$ has a p.d.f (6) in which $\Delta_1(z)$ as defined in Result 3.2.*

Proof. For $x > 0$, the p.d.f of $f_1^{(3)}(x)$ of Z_3 is

$$\begin{aligned} f_1(z) &= f_1(z; \lambda, \beta, \alpha, \gamma)|\frac{dx}{dz}| + f_1(-z; \lambda, \beta, \alpha, \gamma)|\frac{dx}{dz}| \\ &= \frac{\phi(z)}{\gamma+\beta}\left[\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(z))\right] + \frac{\phi(-z)}{\gamma+\beta}\left[\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(-z))\right] \\ &= \frac{\phi(z)}{\gamma+\beta}\left[2\gamma + \beta[\Phi(\alpha)]^{-1}\{\Phi(\theta(z)) + \phi(\theta(-z))\}\right] \\ &= \frac{\phi(z)}{\gamma+\beta}\left[2\gamma + \beta[\Phi(\alpha)]^{-1}\Delta_1(z)\right] \qquad (6) \end{aligned}$$

□

Proposition 3.4. *The cumulative distribution function (c.d.f) $F_1(x)$ of AGGCND$(\lambda, \beta, \alpha, \gamma)$ with p.d.f (4) is the following, for $x \in R$.*

$$F(x) = \frac{\phi(x)}{\gamma+\beta}\left[\gamma + \frac{\beta}{2}[\Phi(\alpha)]^{-1}\right] - \frac{\beta[\Phi(\alpha)]^{-1}}{\gamma+\beta}\xi_0(x, \theta(t)) \qquad (7)$$

where $\xi_0(x, \theta(t)) = \int_x^\infty \int_0^{\theta(t)} \phi(t)\phi(u)\, du dt$, which can be evaluated using the softwares such as MATHCAD, MATHEMATICA etc.

Proof.

$$
\begin{aligned}
f_1(x) &= \int_{-\infty}^{x} f_1(t;\lambda,\beta,\alpha,\gamma)dt \\
&= \frac{\gamma}{\gamma+\beta}\Phi(x) + \frac{\beta[\Phi(\alpha)]^{-1}}{\gamma+\beta}\left[\frac{\Phi(x)}{2} - \int_{x}^{\infty}\int_{0}^{\theta(t)} \phi(t)\phi(u)dudt\right] \\
&= \frac{\Phi(x)}{\gamma+\beta}\left[\gamma + \frac{\beta}{2}[\Phi(\alpha)]^{-1}\right] - \frac{\beta[\Phi(\alpha)]^{-1}}{\gamma+\beta}\xi_0(x,\theta(t))
\end{aligned}
$$

□

Now we derive the characteristic function of AGGCND$(\lambda,\beta,\alpha,\gamma)$. For that we need the following lemma which is taken from Ellison (1964).

Lemma 3.5. *For a standard normal random variable X_1 with distribution function $\Phi(.)$, we have the following for all $a, b \in R$*

$$E\left\{\Phi(aX+b)\right\} = \Phi\left\{\frac{b}{\sqrt{1+a^2}}\right\}$$

Proposition 3.6. *The characteristic function $\phi_{X_1}(t)$ of AGGCND$(\lambda,\beta,\alpha,\gamma)$ with p.d.f (4) is the following, for $t \in R$ and $i = \sqrt{-1}$.*

$$\phi_X(t) = \frac{e^{\frac{-t^2}{2}}}{\gamma+\beta}\left[\gamma + \beta[\Phi(\alpha)]^{-1}E[\Phi(\theta(u+it))]\right] \tag{8}$$

Proof. Let X_1 follows AGGCND$(\lambda,\beta,\alpha,\gamma)$ with p.d.f (4). Then by the definition of characteristic function, we have the following for any $t \in R$ and $i = \sqrt{-1}$

$$
\begin{aligned}
\phi_{X_1}(t) &= E(e^{itX}) \\
&= \frac{\gamma}{\gamma+\beta}\int_{-\infty}^{\infty} e^{itx}\phi(x)dx + \frac{\beta[\Phi(\alpha)]^{-1}}{\gamma+\beta}\int_{-\infty}^{\infty} e^{itx}\phi(x)\Phi(\theta(x))dx \\
&= \frac{e^{\frac{-t^2}{2}}}{\gamma+\beta}\left\{\gamma + \beta[\Phi(\alpha)]^{-1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-(x-it)^2}{2}}\Phi(\theta(x))dx\right\} \qquad (9)
\end{aligned}
$$

On substituting $x - it = u$, in (9) we obtain

$$\phi_{X_1}(t) = \frac{e^{\frac{-t^2}{2}}}{\gamma+\beta}\left[\gamma + \beta[\Phi(\alpha)]^{-1}E[\Phi(\theta(u+it))]\right]$$

which implies (8). □

4. Moments

The expression for even moments and odd moments of AGGCND$(\lambda,\beta,\alpha,\gamma)$ are obtained through the following results.

Proposition 4.1. *If X_1 follows $AGGCND(\lambda, \beta, \alpha, \gamma)$, then for k=1,2,...,*

$$E(X^{2k}) = \frac{2^{k+\frac{1}{2}}}{(\gamma+\beta)\sqrt{2\pi}}\Gamma(k+\frac{1}{2}) + \frac{\beta[\Phi(\alpha)]^{-1}}{2(\gamma+\beta)}A_k(\lambda,\beta), \tag{10}$$

in which

$$A_k(\lambda,\beta) = \int_0^\infty u^{k-\frac{1}{2}}\phi(\sqrt{u})\Phi\left(\theta(\sqrt{u})\right)du,$$

which can be easily evaluated by using the softwares such as MATHCAD, MATHEMATICA etc.

Proof. By the definition of raw moments, for any $k \geq 0$,

$$E(X_1^{2k}) = \int_{-\infty}^{\infty} x^{2k} f_1(x;\lambda,\alpha,\beta)dx. \tag{11}$$

On substituting $x^2 = u$ in (11) we obtain the following ,

$$\begin{aligned} E(X_1^{2k}) &= \frac{1}{\gamma+\beta}\int_0^\infty u^k\phi(\sqrt{u})\frac{1}{\sqrt{u}}du + \frac{\beta[\Phi(\alpha)]^{-1}}{2(\gamma+\beta)} \\ &\quad \int_0^\infty u^k\phi(\sqrt{u})\Phi\left(\theta(\sqrt{u})\right)\frac{1}{\sqrt{u}}du \\ &= \frac{1}{(\gamma+\beta)}\left[\int_0^\infty u^{k-\frac{1}{2}}\phi(\sqrt{u})du + \frac{\beta[\Phi(\alpha)]^{-1}}{2}\right. \\ &\quad \left. u^{k-\frac{1}{2}}\phi(\sqrt{u})\Phi\left(\theta(\sqrt{u})\right)\right]du, \end{aligned}$$

which leads to (10). □

Proposition 4.2. *If X_1 follows $AGGCND(\lambda, \alpha, \beta)$, then for k=0,1,2,...,*

$$E(X_1^{2k+1}) = \frac{2^{k+1}}{(\gamma+\beta)\sqrt{2\pi}}\Gamma(k+1) + \frac{\beta[\Phi(\alpha)]^{-1}}{2(\gamma+\beta)}B_k(\lambda,\beta), \tag{12}$$

in which

$$B_k(\lambda,\beta) = \int_0^\infty u^k\phi(\sqrt{u})\Phi\left(\theta(\sqrt{u})\right)du,$$

which can be easily evaluated using the softwares such as MATHCAD, MATHEMATICA etc.

Proof. By definition of raw moments,

$$E(X_1^{2k+1}) = \int_{-\infty}^{\infty} x^{2k+1} f(x;\lambda,\beta,\alpha,\gamma)dx. \tag{13}$$

On substituting $x^2 = u$ in (13) we get,

$$
\begin{aligned}
E(X_1^{2k+1}) &= \frac{1}{\gamma+\beta}\int_0^\infty u^{k+\frac{1}{2}}\phi(\sqrt{u})\frac{1}{\sqrt{u}}du + \frac{\beta[\Phi(\alpha)]^{-1}}{2(\gamma+\beta)} \\
&\quad \int_0^\infty u^{k+\frac{1}{2}}\phi(\sqrt{u})\Phi\left(\theta(\sqrt{u})\right)\frac{1}{\sqrt{u}}du \\
&= \frac{1}{(\gamma+\beta)}\left[\int_0^\infty u^k\phi(\sqrt{u})du + \frac{\beta[\Phi(\alpha)]^{-1}}{2}\right. \\
&\quad \left.\int_0^\infty u^k\phi(\sqrt{u})\Phi\left(\theta(\sqrt{u})\right)\right]du,
\end{aligned}
$$

which leads to (12). □

5. Reliability measures and Mode

Here we investigate some properties of AGGCND$(\lambda, \beta, \alpha, \gamma)$ with p.d.f. (4) useful in reliability studies.

Let X_1 follows AGGCND$(\lambda, \beta, \alpha, \gamma)$ with p.d.f (4). Now from the definition of reliability function $R(t)$, failure rate $r(t)$ and mean residual life function $\mu(t)$ of X_1 we obtain the following results.

Proposition 5.1. *The reliability function $R(t)$ of X_1 is the following, in which $\xi_0(t, \theta(x))$ is as defined in Result 3.4.*

$$R(t) = \frac{1}{\gamma+\beta}\left[1-\Phi(t)\right]\left\{\gamma + \frac{\beta[\Phi(\alpha)]^{-1}}{2}\right\} + \frac{\beta[\Phi(\alpha)]^{-1}}{\gamma+\beta}\xi_0(t, \theta(x))$$

Proposition 5.2. *The failure rate $r(t)$ of X_1 is given by,*

$$r(t) = \frac{\phi(t)\left[\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(t))\right]}{(1-\Phi(t))\left[\gamma + \frac{\beta[\Phi(\alpha)]^{-1}}{2}\right] + \beta[\Phi(\alpha)]^{-1}\xi_0(t, \theta(x))}$$

The failure rate plots of the AGGCND$(\lambda, \beta, \alpha, \gamma)$ for different values of γ are plotted given in Figure 2.

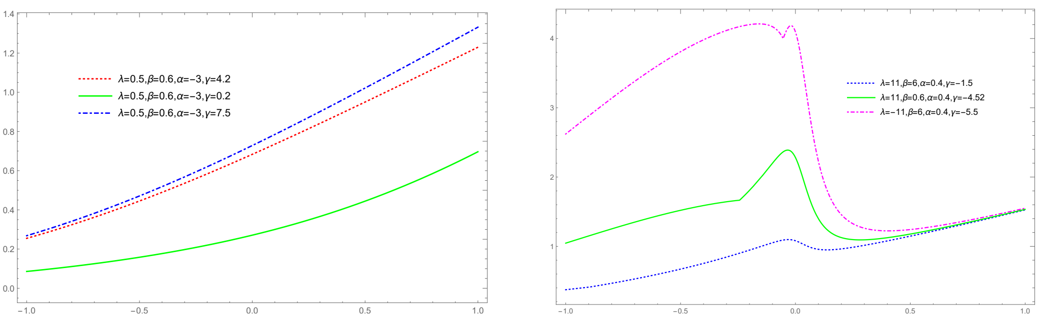

Figure 2. Failure rate plots of the AGGCND$(\lambda, \beta, \alpha, \gamma)$ for fixed values of λ, β, α and various values of γ.

The failure rate plot of the AGGCND$(\lambda,\beta,\alpha,\gamma)$ is presented in Figure 2. From the figure, one can observe that the failure rate plot of the AGGCND$(\lambda,\beta,\alpha,\gamma)$ has an increasing failure rate for $\lambda > 0, \alpha < 0$, $\beta > 0$ and $\gamma > 0$, while it exhibits an increasing and decreasing failure rate function for $\lambda < 0$ and $\alpha < 0$, $\beta < 0$, $\gamma < 0$.

Proposition 5.3. *The mean residual life function of AGGCND$(\lambda,\alpha,\beta,\gamma)$ is*

$$\mu(t) = \frac{1}{(\gamma+\beta)R(t)}\left[\gamma\phi(t) + \beta[\Phi(\alpha)]^{-1}\Phi\left(\theta(t)\right)\phi(t) + \Phi(\alpha)]^{-1}\Lambda_\beta(t;\lambda)\right] - t \qquad (14)$$

where

$$\Lambda_\beta(t;\lambda) = \int_t^\infty \phi(x)\left[\frac{d}{dx}\left(\int_0^{\theta(x)}\phi(u)du\right)\right]dx$$

Proof. By definition, the mean residual life function (MRLF) of X_1 is given by

$$\begin{aligned}\mu(t) &= E(X_1 - t|X_1 > t) \qquad (15)\\ &= E(X_1|X_1 > t) - t,\end{aligned}$$

where

$$\begin{aligned}E(X_1|X_1 > t) &= \frac{\gamma}{R(t)(\gamma+\beta)}\int_t^\infty x\phi(x)dx \qquad (16)\\ &+\frac{\beta[\Phi(\alpha)]^{-1}}{R(t)}\int_t^\infty x\phi(x)\Phi\left(\theta(x)\right)dx.\end{aligned}$$

Since $\phi(.)$ is the p.d.f of standard normal variate $\phi^{'}(x) = -x\phi(x)$. Therefore (16) becomes,

$$\begin{aligned}E(X_1|X_1 > t) &= \frac{\gamma}{(\gamma+\beta)R(t)}\int_t^\infty -\phi^{'}(x)dx \qquad (17)\\ &+\frac{\beta[\Phi(\alpha)]^{-1}}{(\gamma+\beta)R(t)}\int_t^\infty -\phi^{'}(x)\Phi\left(\theta(x)\right)dx.\end{aligned}$$

On integrating (17), we obtain the following

$$\begin{aligned}E(X_1|X_1 > t) &= \frac{\gamma}{(\gamma+\beta)R(t)}\phi(t) + \frac{\beta[\Phi(\alpha)]^{-1}}{(\gamma+\beta)R(t)}\left(-\Phi(\theta(x))\phi(x)\right)_t^\infty\\ &-\frac{\beta[\Phi(\alpha)]^{-1}}{R(t)(\gamma+\beta)}\int\limits_t^\infty -\phi(x)\left[\frac{d}{dx}\left(\int_{-\infty}^{\theta(x)}\phi(u)du\right)\right]dx\end{aligned} \qquad (18)$$

On solving (18) and substituting in (17), we get (14).

The functions $R(t), r(t)$ and $\mu(t)$ are equivalent in the sense that if one of them is given, the other two can be uniquely determined. □

Proposition 5.4. *Case 1: For $x > 0$ the p.d.f of AGGCND$(\lambda,\alpha,\beta,\gamma))$ is log concave*

(i) *if* $\lambda < 0$ *provided for all* $\beta \geq 0, \alpha > 0$ *and*$\gamma > 0$ *and*
(ii) *if* $\lambda > 0$, *provided* $|A_1 + A_2| < |1 + A_3|$,

where A_1, A_2 *and* A_3 *are as defined in* (19), (20) *and* (21).
Case 2: For $x < 0$ *the p.d.f of* $AGGCND(\lambda, \alpha, \beta, \gamma))$ *is log concave*

(i) *if* $\lambda > 0$ *provided for all* $\beta \geq 0, \alpha > 0$ *and* $\gamma > 0$ *and*
(ii) *if* $\lambda < 0$ *provided* $|A_1 + A_2| < |1 + A_3|$.

Proof. To establish $ln[f_1(x; \lambda, \alpha, \beta, \gamma)]$ is a concave function of X_1, it is enough to show that its second derivative is negative for all X_1. Then

$$\frac{d}{dx}\{ln\,[f_1(x; \lambda, \alpha, \beta, \gamma)]\} = -x + \frac{\beta[\Phi(\alpha)]^{-1}\theta^{'}(x)\phi(\theta(x))}{\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(x))}$$

and

$$\frac{d^2}{dx^2}\{ln[f_1(x; \lambda, \alpha, \beta, \gamma)]\} = -1 - A_1 - A_2 + A_3$$

where

$$A_1 = \frac{\beta[\Phi(\alpha)]^{-1}(\theta^{'}(x))^2\theta(x)\phi(\theta(x))}{\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(x))} \tag{19}$$

$$A_2 = \frac{\beta^2[\Phi(\alpha)]^{-2}(\phi(\theta(x)))^2(\theta^{'}(x))^2}{(\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(x)))^2} \tag{20}$$

$$A_3 = \frac{\beta[\Phi(\alpha)]^{-1}\phi(\theta(x))\theta^{''}(x)}{\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(x))} \tag{21}$$

Note that $\Phi(\theta(x))$ and $\phi(\theta(x))$ are positive for all $x \in R$ and hence $A_1 > 0$ for $x > 0$, $\beta > 0$ and $\gamma > 0$ or $x < 0$, $\beta < 0$ and $\gamma < 0$ and $A_3 > 0$ for $\beta > 0, \alpha > 0$ and $\gamma > 0$ or $\beta < 0, \alpha < 0$ and $\gamma < 0$. Clearly $A_2 > 0$ for all values of $\beta, \alpha, \lambda, \gamma$. Also $\gamma + \beta[\Phi(\alpha)]^{-1}\Phi(\theta(x))$ is positive for all values of β, α, γ and λ . Hence (4) is log concave in these situations. □

As a consequence of Result 5.4, we have the following results regarding the unimodality and plurimodality of the $AGGCND(\lambda, \alpha, \beta, \gamma))$.

Proposition 5.5. *AGGCND*($\lambda, \beta, \alpha, \gamma$) *density is strongly unimodal under the following two cases.*
Case 1: For $x > 0$

(i) *if* $\lambda < 0$ *provided for all* $\beta \geq 0, \alpha > 0$ *and*$\gamma > 0$ *and*
(ii) *if* $\lambda > 0$, *provided* $|A_1 + A_2| < |1 + A_3|$,,

Case 2: For $x < 0$ *the p.d.f of* $AGGCND(\lambda, \alpha, \beta, \gamma))$ *is unimodal*

(i) *if* $\lambda > 0$ *provided for all* $\beta \geq 0, \alpha > 0$ *and* $\gamma > 0$ *and*
(ii) *if* $\lambda < 0$ *provided* $|A_1 + A_2| < |1 + A_3|$.

Proposition 5.6. *AGGCND*($\lambda, \beta, \alpha, \gamma$) *density is plurimodal under the following two cases.*
Case 1: For $x > 0$

(i) if $\lambda < 0$ provided for all $\beta > 0, \alpha > 0$ and $\gamma < 0$ and
(ii) if $\lambda > 0$, $\beta > 0, \alpha > 0$ and $\gamma < 0$ provided $|A_1 + A_2| > |1 + A_3|$,

Case 2: For $x < 0$ the p.d.f of $AGGCND(\lambda, \alpha, \beta, \gamma))$ is plurimodal

(i) if $\lambda < 0$ provided for all $\beta < 0, \alpha < 0$ and $\gamma < 0$ and
(ii) if $\lambda > 0$, $\beta > 0$ and $\gamma > 0$ provided $|A_1 + A_2| > |1 + A_3|$.

6. Extended AGGCND

In this section we discuss an extended form of AGGCND$(\lambda, \alpha, \beta, \gamma)$ by introducing the location parameter μ and scale parameter σ.

Definition 6.1. Let $X_1 \sim AGGCND(\lambda, \alpha, \beta, \gamma))$ with p.d.f given in (4). Then $Y_1 = \mu + \sigma X_1$ is said to have an extended AGGCND with $\mu, \sigma, \lambda, \beta, \alpha$ and γ with the following p.d.f

$$f_1^*(y, \mu, \sigma; \lambda, \alpha, \beta, \gamma) = \frac{1}{\sigma(\gamma + \beta)} \phi \left(\frac{y - \mu}{\sigma} \right) \left[\gamma + \beta [\Phi(\alpha)]^{-1} \Phi \left(\theta^*(y) \right) \right], \tag{22}$$

where $\theta^*(y) = \alpha\sqrt{1 + \lambda} + \frac{\lambda(y-\mu)}{\sqrt{\sigma^2+\lambda^2(y-\mu)^2}}$ in which $y \in R$, $\mu \in R$, $\sigma > 0$, $\lambda \in R$, $\gamma \in R$, $\alpha \in R$ and $\beta > -1$. A distribution with p.d.f (22) is denoted as EAGGCND$(\mu, \sigma; \lambda, \alpha, \beta, \gamma)$.

Now we have the following results. The proof of these results are similar to the results given in $AGGCND(\lambda, \alpha, \beta, \gamma)$ and hence omitted.

Proposition 6.2. *The c.d.f $f_1(x)$ of EAGGCND$(\mu, \sigma; \lambda, \alpha, \beta, \gamma)$ with p.d.f (22) is the following, for $y \in R$.*

$$F^*(y) = \frac{\Phi\left(\frac{y-\mu}{\sigma}\right)}{\sigma(\gamma + \beta)} \left[\gamma + \frac{\beta}{2} [\Phi(\alpha)]^{-1} \right] - \frac{\beta[\Phi(\alpha)]^{-1}}{\sigma(\gamma + \beta)} \xi_0^*(y, \theta^*(t))$$

where $\xi_0^*(y, \theta^*(t))$ is as defined in Result 3.4.

Proposition 6.3. *The characteristic function of EAGGCND$(\mu, \sigma; \lambda, \alpha, \beta, \gamma)$ is given by*

$$\psi_{Y_1}(t) = \frac{1}{\sigma(\gamma + \beta)} e^{it\mu - \frac{t^2\sigma^2}{2}} \left\{ \gamma + \beta [\Phi(\alpha)]^{-1} E \left[\Phi(\theta^*(z)) \right] \right\},$$

where $\theta^*(z) = \beta\sqrt{1 + \lambda} + \frac{\lambda(z+\sigma^2 it)}{\sqrt{\sigma^2+\lambda^2(z+\sigma^2 it)^2}}$

Proposition 6.4. *The reliability function $R^*(t)$ of Y is the following, in which $\xi_0^*(t, \theta^*(y))$ is as defined in Result 3.4.*

$$R^*(t) = \frac{1}{\gamma + \beta} \left[1 - \Phi\left(\frac{t - \mu}{\sigma} \right) \right] \left\{ \gamma + \frac{\beta[\Phi(\alpha)]^{-1}}{2} \right\} + \frac{\beta[\Phi(\alpha)]^{-1}}{\gamma + \beta} \xi_0^*(t, \theta^*(y))$$

Proposition 6.5. *The failure rate $r^*(t)$ of Y_1 is given by*

$$r^*(t) = \frac{\phi\left(\frac{t-\mu}{\sigma}\right)\left[\gamma + \beta[\Phi(\alpha)]^{-1}\Phi\left(\theta^*(t)\right)\right]}{\left[1-\Phi(\frac{t-\mu}{\sigma})\right]\left\{\gamma + \frac{\beta[\Phi(\alpha)]^{-1}}{2}\right\} + \beta[\Phi(\alpha)]^{-1}\xi_0^*(t,\theta^*(y))}.$$

7. Maximum likelihood estimation

The log likelihood function, ln L of the random sample of size n from a population following EAGGCND$(\mu, \sigma; \lambda, \alpha, \beta, \gamma)$ is the following in which $c = -\frac{n}{2}ln\,2\pi$

$$\begin{aligned} ln\,L &= c - nln(\gamma+\beta) - nln\sigma - \frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma^2} \\ &\quad + \sum_{i=1}^{n} ln\left(\gamma + \beta\left[\Phi(\alpha)\right]^{-1}\Phi\left(\theta^*(y_i)\right)\right) \end{aligned} \tag{23}$$

On differentiating (23) with respect to parameters $\mu, \sigma, \lambda, \beta, \alpha$ and γ and then equating to zero, we obtain the following normal equations.

$$\sum_{i=1}^{n}\frac{(y_i-\mu)}{\sigma^2} - \beta[\Phi(\alpha)]^{-1}\sum_{i=1}^{n}\frac{\phi\left(\theta^*(y_i)\right)\left(\frac{\lambda^3(y_i-\mu)^2}{[\sigma^2+\lambda^2(y_i-\mu)^2]^{\frac{3}{2}}} - \frac{\lambda}{\sqrt{\lambda^2(y_i-\mu)^2+\sigma^2}}\right)}{\gamma + \beta[\Phi(\alpha)]^{-1}\Phi\left(\theta^*(y_i)\right)} = 0 \tag{24}$$

$$\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma^3} - \beta[\Phi(\alpha)]^{-1}\sum_{i=1}^{n}\frac{\phi\left(\theta^*(y_i)\right)(y_i-\mu)\left(\frac{\lambda(y_i-\mu)\sigma}{(\sigma^2+\lambda^2(y_i-\mu)^2)^{\frac{3}{2}}}\right)}{\gamma + \beta[\Phi(\alpha)]^{-1}\Phi\left(\theta^*(y_i)\right)} = \frac{n}{\sigma} \tag{25}$$

$$\beta[\Phi(\alpha)]^{-1}\sum_{i=1}^{n}\frac{\phi\left(\theta^*(y_i)\right)\left[\frac{\alpha\lambda}{\sqrt{1+\lambda^2}} - \frac{\lambda^2(y_i-\mu)^3}{(\sigma^2+\lambda^2(y_i-\mu))^{\frac{3}{2}}}\frac{y_i-\mu}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right]}{\gamma + \beta[\Phi(\alpha)]^{-1}\Phi\left(\theta^*(y_i)\right)} = 0 \tag{26}$$

$$[\Phi(\alpha)]^{-1}\sum_{i=1}^{n}\frac{\Phi\left(\theta^*(y_i)\right)}{\gamma + \beta[\Phi(\alpha)]^{-1}\Phi\left(\theta^*(y_i)\right)} = \frac{n}{\gamma+\beta} \tag{27}$$

$$\beta[\Phi(\alpha)]^{-1}\sum_{i=1}^{n}\left[\frac{\phi(\theta^*(y_i))\sqrt{1+\lambda^2} - \Phi(\theta^*(y_i))[\Phi(\alpha)]^{-1}\phi(\alpha)}{\gamma + \beta[\Phi(\alpha)]^{-1}\Phi\left(\theta^*(y_i)\right)}\right] = 0 \tag{28}$$

and

$$\sum_{i=1}^{n}\frac{1}{\gamma + \beta[\Phi(\alpha)]^{-1}\Phi\left(\theta^*(y_i)\right)} = \frac{n}{\gamma+\beta}. \tag{29}$$

On solving the equations (24) to (29) we get the maximum likelihood estimate(MLE) of the parameters of EAGGCND$(\mu, \sigma; \lambda, \alpha, \beta, \gamma)$.

8. Data Illustration

In this section we consider a real life data applications of the EAGGCND. The data set is taken from Cook and Weisberg (1994) which is based on the body mass index (BMI) values for the 50 females and is given below Data set 1:
24.47 23.99 26.24 20.04 25.72 25.64 19.87 23.35 22.42 20.42 22.13 25.17 23.72
21.28 20.87 19.00 22.04 20.12 21.35 28.57 26.95 28.13 26.85 25.27 31.93 16.75
19.54 20.42 22.76 20.12 22.35 19.16 20.77 19.37 22.37 17.54 19.06 20.30 20.15
25.36 22.12 21.25 20.53 17.06 18.29 18.37 18.93 17.79 17.05 20.31.
For illustrating the suitability of the model, we have fitted EAGGCND$(\mu, \sigma; \lambda, \beta, \alpha, \gamma)$ to the above data set and computed the Kolmogorov Smirnov Statistic (KSS) values, Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike's Information Criterion (AICc) values. All these numerical results obtained are presented in Table 1.

Table 1. Estimated values of the parameters for the model: EAGGCND$(\mu, \sigma; \lambda, \beta, \alpha, \gamma)$, EACND $(\mu, \sigma; \lambda, \beta, \alpha)$ and EGSCND$(\mu, \sigma; \lambda, \beta, \alpha, \gamma)$ with respective values of KSS, AIC, BIC and AICc.

Data sets	Estimates of the parameters	ESCND$(\mu, \sigma; \lambda, \beta)$	EACND $(\mu, \sigma; \lambda, \beta, \alpha)$	EAGGCND$(\mu, \sigma; \lambda, \beta, \alpha, \gamma)$
1	$\hat{\mu}$	21.01	20.86	21.6
	$\hat{\sigma}$	3.3004	3.33	3.3
	$\hat{\lambda}$	10.001	8.231	6.36
	$\hat{\beta}$	2.197	0.562	0.21
	$\hat{\alpha}$	-	19.314	5.05
	$\hat{\gamma}$	-	-	9.3
	KSS Statistic	0.182	0.139	0.09
	P-value	0.062	0.261	0.777
	AIC	278.349	275.938	273.622
	BIC	285.997	285.498	284.094
	AICc	279.238	277.302	275.575

From Table 1, it is clear that the EAGGCND$(\mu, \sigma; \lambda, \alpha, \beta, \gamma)$ is a more appropriate model to all the data sets considered in this paper compared to the existing models ESCND$(\mu, \sigma; \lambda, \beta)$, EACND $(\mu, \sigma; \lambda, \beta, \alpha)$. Thus, the model discussed in this paper provides more flexibility in modeling perspectives due to the presence of extra parameter. Also, we have plotted the histogram of data set 1 along with the fitted probability plots corresponding to the $ESCND, EACND$ and $EAGGCND$ in the figure 3. From the figure it can be seen that the $EAGGCND$ yields a better fit compared to the existing models $ESCND$ and $EACND$ in case of the above data set.

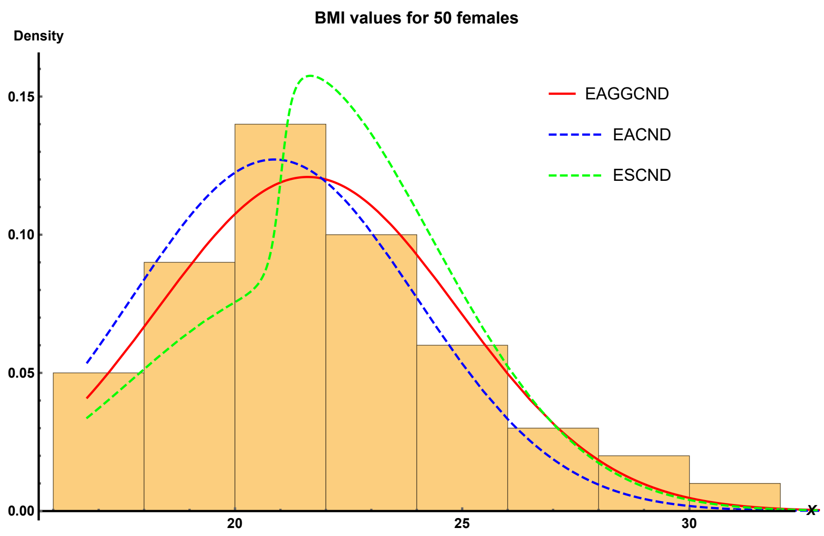

Figure 3. Histogram of Data set 1 and fitted distributions

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9. Generalized likelihood ratio test (GLRT)

In this section, we discuss the generalized likelihood ratio test (GLRT) procedure for testing the significance of the additional parameter of the model.

(1) $H_0^{(1)} : \lambda_2 = 0$
(2) $H_0^{(2)} : \lambda_2 = 0$ and $\beta = 0$
(3) $H_0^{(3)} : \alpha = 0$ and $\beta = 0$.

The test statistic is

$$-2ln\Lambda(x) = 2[lnL(\hat{\Theta}; x) - lnL(\hat{\Theta}^*; x)], \tag{30}$$

where $\hat{\Theta}$ is the maximum likelihood estimator of $\Theta = (\mu, \sigma; \lambda, \alpha, \beta, \gamma)$ with no restrictions, and $\hat{\Theta}^*$ is the maximum likelihood estimator for Θ under the null hypothesis H_0. The test statistic given in (30) follows chi-square distribution with 1 degrees of freedom (d.f) for those hypotheses having one parameter restriction and two d.f for those hypotheses having two parameter restrictions. The results based on GLRT are given in Table 2.

Table 2. Computed values of $lnL(\hat{\Theta}; x), lnL(\hat{\Theta}^*; x)$, GLRT statistics and P-value of EAGGCND

Data set	$lnL(\hat{\Theta}^*; x)$	$lnL(\hat{\Theta}; x)$	GLRT	d.f	Chi-square value	P-value
Test 1	-133.328	-130.811	5.033	1	3.85	0.024
Test 2	-135.175	-130.811	8.727	2	5.99	0.0127

Now by adopting the test procedure discussed in section 9, we test $H_0 : \gamma = 2$ against the alternative hypothesis $H_1 : \gamma \neq 2$ against the alternative hypothesis $H_1 : \alpha \neq 0$ and $\gamma \neq 2$. The numerical results obtained are given in Table 2. Based on the computed values of GLRT and its P-value from Table 2, one can observe that the null hypothesis is rejected in the case of the above data sets, which indicates the suitability of the model EAGGCND to the dataset considered in this paper.

10. Simulation Study

In order to assess the performance of the maximum likelihood estimators of the parameters of the EAGGCND$(\mu, \sigma; \lambda_1, \lambda_2, \alpha, \beta, \gamma)$, we have conducted a brief simulation study by generating observations with the help of MATHEMATICA for the following sets of parameters $\mu = 21.6, \sigma = 3.3, \lambda = 6.36, \beta = 0.21$, $\alpha = 5.05$ and $\gamma = 9.3$. We have considered 200 bootstrap samples of sizes 20, 50, 100 and 500 from the EAGGCND for comparing the performances of the maximum likelihood estimators. The likelihood estimates of the parameters, the average bias estimates and average MSEs over 200 replications are calculated and presented in Table 3.

Table 3. Estimates of the parameters and corresponding bias and MSEs of $EGAMND$ based on simulated data sets corresponding to parameter set $\mu = 21.6, \sigma = 3.3, \lambda = 6.36, \beta = 0.21$, $\alpha = 5.05$ and $\gamma = 9.3$.

Simulated Data Sets	Sample size	Parameter Set	Estimate	Bias	MSE
(1)	20	$\hat{\mu}$	21.575	-0.024	0.00059
		$\hat{\sigma}$	3.12	-0.178	0.0318
		$\hat{\lambda}$	6.36	0.009	8.1e-05
		$\hat{\beta}$	0.29	0.08	0.0064
		$\hat{\alpha}$	5.089	0.039	0.0015
		$\hat{\gamma}$	9.39	0.089	0.008
	50	$\hat{\mu}$	21.585	-0.014	0.00019
		$\hat{\sigma}$	3.179	] -0.12	0.0141
		$\hat{\lambda}$	6.367	0.007	4.9e-05
		$\hat{\beta}$	0.27	0.06	0.0036
		$\hat{\alpha}$	5.069	0.019	0.00039
		$\hat{\gamma}$	9.37	0.069	0.0048
	100	$\hat{\mu}$	21.589	-0.01	0.0001
		$\hat{\sigma}$	3.278	-0.021	0.00047
		$\hat{\lambda}$	6.364	0.004	1.6e-05
		$\hat{\beta}$	0.24	0.03	0.0009
		$\hat{\alpha}$	5.039	-0.01	0.0001
		$\hat{\gamma}$	9.34	0.039	0.0015
	500	$\hat{\mu}$	21.592	-0.007	4.944e-05
		$\hat{\sigma}$	3.296	-0.0033	1.127e-05
		$\hat{\lambda}$	6.359	-0.001	1e-06
		$\hat{\beta}$	0.22	0.01	0.0001
		$\hat{\alpha}$	5.05	0.0089	8.098e-05
		$\hat{\gamma}$	9.29	-0.01	0.0001

From Table 3, it can be observed that both the bias and MSE are in decreasing order as sample size increases.

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